Path by path uniqueness of stochastic differential equations

Lukas Wresch

Bielefeld University

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On a fixed probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ we consider the stochastic differential equation

$$\mathrm{d}X_t = -AX_t\mathrm{d}t + f(t, X_t)\mathrm{d}t + \mathrm{d}W_t, \qquad X_0 = 0 \qquad \text{(SDE)}.$$

- W is a cylindrical **Brownian motion**,
- $f : [0, T] \times H \longrightarrow H$ is a **bounded** Borel function,
- $A\colon D(A) \longrightarrow H$ is positive definite, self-adjoint, linear, A^{-1} is trace class.

$$Ae_n = \lambda_n e_n, \qquad \text{with } \lambda_n > 0 \text{ and}$$

$$\sum_{n\in\mathbb{N}} \lambda_n^{-1} < \infty.$$

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Solution

We say a **stochastic process** $(X_t)_{t\in[0,T]}$ is a **solution** to

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if $\mathbb{P}$-a.s. for all $t \in [0, T]$ we have

$$X_t = \int_0^t e^{-(t-s)A} f(s, X_s)\,\mathrm{d}s + \underbrace{\int_0^t e^{-(t-s)A}\,\mathrm{d}W_s}\,. \qquad \text{(IE)}$$

I.e. a **mild solution**.

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I.e. a **mild solution**.

Pathwise Uniqueness

Let $(X_t)_{t\in[0,T]}$ and $(Y_t)_{t\in[0,T]}$ be two solutions then $\mathbb{P}$-a.s. $X_t = Y_t$ for all $t \in [0, T]$, i.e.

$$\exists\Omega_0 \subseteq \Omega \colon \mathbb{P}[\Omega_0] = 1 \colon \forall(\omega, t) \in \Omega_0 \times [0, T] \colon X_t(\omega) = Y_t(\omega).$$

However, Ω_0, a priori, **depends** on both X and Y.

Question: Can Ω_0 be chosen **independently** of X and Y?
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We first plug in an $\omega \in \Omega$ into the corresponding integral equation of the mild form of (SDE)

$$x_t = \int_0^t e^{-(t-s)A} f(s, x_s)\, \mathrm{d}s + Z_t^A(\omega). \qquad (\mathrm{IE}_\omega)$$

Aim: For **fixed** $\omega \in \Omega$, find a **unique** continuous **function** $x\colon [0, T] \longrightarrow H$ satisfying the equation above.

Uniqueness for ODEs in integral form perturbed by an Ornstein–Uhlenbeck **path** $Z^A(\omega)$.

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Path-by-Path Uniqueness

Definition

Fix $\omega \in \Omega$. Denote by $\mathcal{S}(\omega)$ the set of all **functions** x for which

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holds.

Definition (Path-by-Path Uniqueness)

We say (SDE) exhibits **path-by-path uniqueness** if there exists a measurable set $\Omega_0 \subseteq \Omega$ with $\mathbb{P}[\Omega_0] = 1$ such that

$$\#\mathcal{S}(\omega) \leq 1, \qquad \forall \omega \in \Omega_0.$$

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Path-by-Path Uniqueness VS. Pathwise Uniqueness

Path-by-path uniqueness: There is a null set $N \subseteq \Omega$ such that **all solutions** coincide on $\Omega \setminus N$. (Uniqueness in the sense of random ODEs).

Pathwise uniqueness: Let X, Y be two solutions. Then $X = Y$ on $\Omega \setminus N_{X,Y}$. The null set **depends** on the solutions!

$$\text{Path-by-path uniqueness} \Longrightarrow \text{Pathwise uniqueness}$$

In the case $H := \mathbb{R}^d$ and $A = 0$ path-by-path uniqueness was proved by A. Davie in 2007.
The case $A = -\frac{d^2}{dx^2}$ was proved by O. Butkovsky and L. Mytnik in 2016.

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Strategy of the proof

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$$\mathrm{d}Y_t = -AY_t\,\mathrm{d}t + f(t, Y_t)\,\mathrm{d}t + \mathrm{d}W_t$$

and

$$Y_t \text{ is } \mathcal{F}_t\text{-adapted} \quad (W_t \text{ is a cylindrical } \mathcal{F}_t\text{-Brownian motion}).$$

Using **Girsanov's Transformation** we construct an equivalent measure $\mathbb{Q} \approx \mathbb{P}$ (and a cylindrical Brownian motion $\tilde{W}$) such that Y is an Ornstein–Uhlenbeck process, i.e.

$$\mathrm{d}Y_t = -AY_t\,\mathrm{d}t + \mathrm{d}\tilde{W}_t$$

under $\mathbb{Q} \approx \mathbb{P}$.

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The reduced problem

Fix $\omega \in \Omega$. Let x be a solution to (IE$_\omega$). Set $u := x - Y(\omega)$. Then

$$u(t) = \int_0^t e^{-(t-s)A} f(s, \underbrace{x_s}_{=u(s)+Y_s(\omega)}) - e^{-(t-s)A} f(s, Y_s(\omega))\, \mathrm{d}s.$$

Since Y is an Ornstein–Uhlenbeck process under $\mathbb{Q} \approx \mathbb{P}$ we have reduced our problem to:

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for all $\omega \in \Omega_0 \subseteq \Omega$, $\mathbb{Q}[\Omega_0] = 1$.

The reduced problem

Fix $\omega \in \Omega$. Let x be a solution to (IE_ω). Set $u := x - Y(\omega)$. Then

$$u(t) = \int_0^t e^{-(t-s)A} f(s, \underbrace{x_s}_{=u(s)+Y_s(\omega)}) - e^{-(t-s)A} f(s, Y_s(\omega))\, \mathrm{d}s.$$

Since Y is an Ornstein–Uhlenbeck process under $\mathbb{Q} \approx \mathbb{P}$ we have reduced our problem to:

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Regularizing Noise

Let $(X_t, \mathcal{F}_t)$ be a stochastic process.

$$\mathbb{P}\left[|\varphi_{s,t}(x,y)|_H > \eta |t-s|^h |x-y|_H \,\middle|\, \mathcal{F}_s \right] \leq C e^{-c\eta^\alpha}. \qquad (*)$$

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$$\varphi_{s,t}(x,y) := \int_s^t f(r, x + X_r(\omega)) - f(r, y + X_r(\omega)) \, \mathrm{d}r.$$

Definition

Given $Q \subseteq H$. If $(*)$ holds for **every** $f : [0,1] \times H \longrightarrow Q$ and $x, y \in Q$ then we say X is a Q-**regularizing noise** with index h and order α.

A. Davie 2007: Brownian motion in $\mathbb{R}^d$ is regularizing with $h = \frac{1}{2}$, $\alpha = 2$
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Let $Q \subseteq H \subseteq \mathbb{R}^{\mathbb{N}}$. For every $m \in \mathbb{N}$ consider $Q \cap 2^{-m}\mathbb{Z}^{\mathbb{N}}$.

$$x \in Q \cap 2^{-m}\mathbb{Z}^{\mathbb{N}} \qquad \text{then} \qquad x = (*, \dots, *, 0, \dots).$$

Let $d_m \in \mathbb{N} \cup \{+\infty\}$ be the **smallest** number such that $x_n = 0$ for all $n \geq d_m$ and all $x \in Q \cap 2^{-m}\mathbb{Z}^{\mathbb{N}}$.

Definition

For $Q \subseteq H \subseteq \mathbb{R}^{\mathbb{N}}$ we define the **effective dimension** of Q as

$$\mathrm{ed}(Q) := (d_m)_{m \in \mathbb{N}}$$

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Properties of $\varphi_{n,k}$

Recall that

$$\varphi_{n,k}(x,\omega) := \int_{k2^{-n}}^{(k+1)2^{-n}} f(t, x + X_t(\omega)) - f(t, X_t(\omega)) \, \mathrm{d}t.$$

If X is a Q-regularizing noise and $\mathrm{ed}(Q)_m \lesssim \ln(m+1)^{1/\gamma}$ we have

$$|\varphi_{n,k}(x)|_H \leq C n^{\frac{1}{\alpha}+\frac{1}{\gamma}} 2^{-hn} \left(|x|_H + 2^{-2^n}\right)$$

and

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Theorem (Approximation Theorem)

Let $h_n \colon [0,1] \longrightarrow H$ be a sequence of Lipschitz continuous functions converging pointwise to a Lipschitz function h, then

$$\int_0^1 f(t, X_t(\omega) + h_n(t))\, \mathrm{d}t \overset{n\to\infty}{\longrightarrow} \int_0^1 f(t, X_t(\omega) + h(t))\, \mathrm{d}t.$$

Approximation Theorem

$$\int_0^1 f(t, X_t(\omega) + h_n(t))\,\mathrm{d}t \xrightarrow{n\to\infty} \int_0^1 f(t, X_t(\omega) + h(t))\,\mathrm{d}t.$$

If f were continuous this would follow from Lebesgue's dominated convergence Theorem. We approximate f by a continuous function $\overline{f}$. We are left with proving that

$$\int_0^1 \mathbb{1}_{\{f\neq\overline{f}\}}(t, X_t(\omega) + h(t))\,\mathrm{d}t \leq \varepsilon$$

uniformly in h. We can construct $\overline{f}$ such that $\{f \neq \overline{f}\}$ is open, so that by exploiting the lower semi-continuity we have

$$\int_0^1 \mathbb{1}_{\{f\neq\overline{f}\}}(t, X_t(\omega) + h(t))\,\mathrm{d}t \leq \lim_{n\to\infty} \int_0^1 \mathbb{1}_{\{f\neq\overline{f}\}}(t, X_t(\omega) + h_n(t))\,\mathrm{d}t.$$

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We rewrite the limit as a telescoping sum

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where m will be chosen later. We split the second integral into the dyadic intervals $[k2^{-(n+1)}, (k+1)2^{-(n+1)}[$.

$$\int_0^1 \mathbb{1}_{\{f\neq\bar{f}\}}(t, X_t(\omega) + h_m(t))\,\mathrm{d}t$$

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So that we are left with estimating

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Let

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- X a Q-regularizing noise,
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Then there exists a measurable set $\Omega_0 \subseteq \Omega$ with $\mathbb{P}[\Omega_0] = 1$ such that

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for all $\omega \in \Omega_0$.

Main result (in abstract form)

Let

- $Q \subseteq H$ such that $\mathrm{ed}(Q)_m \lesssim \ln(m+1)^{1/\gamma}$, $\gamma > 2$,
- X a Q-regularizing noise,
- $f\colon [0,1] \times H \longrightarrow Q$.

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Main result

Let

$$Q := \{x \in \mathbb{R}^{\mathbb{N}} \colon \sum_{n\in\mathbb{N}} \lambda_n e^{2\lambda_n} |x_n|^2 < C\} \cap \{x \in \mathbb{R}^{\mathbb{N}} \colon |x_n| < \exp(-e^{cn^\gamma})\} \qquad \gamma > 2$$

$$X := Z^A.$$

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Assumptions on the drift $f = (f_n)_{n\in\mathbb{N}}$

We assume that

- $\|f\|_{\infty,A} := \sup_{t\in[0,T],x\in H} \sum_{n\in\mathbb{N}} \lambda_n e^{2\lambda_n}|f_n(t,x)|^2 < \infty$
- $\|f_n\|_\infty = \sup_{t\in[0,T],x\in H} |f_n(t,x)| \leq \exp\left(-e^{n^\gamma}\right)$, with $\gamma > 2$

Main result

Theorem (Main result, LW17)

Under the above assumptions path-by-path uniqueness holds for equation (SDE).

Idea of the proof of the main result

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Fix an $\omega \in \Omega$. Let u be a function solving

$$u(t) = \int_0^t e^{-(t-s)A} \left(f(s, u(s) + X_s(\omega)) - f(s, X_s(\omega)) \right) \, \mathrm{d}s$$

We have to show that $u \equiv 0$. For $n \in \mathbb{N}$ let $k \in \{0, ..., 2^n - 1\}$. We have

$$|u((k+1)2^{-n}) - u(k2^{-n})|_H$$

$$\approx \left| \int_{k2^{-n}}^{(k+1)2^{-n}} e^{-((k+1)2^{-n}-s)A} \left(f(s, u(s) + X_s(\omega)) - f(s, X_s(\omega)) \right) \, \mathrm{d}s \right|_H$$

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$$|u((k+1)2^{-n}) - u(k2^{-n})|_H \approx |\varphi_{n,k}(u)|_H$$

Let u_ℓ be a sequence of functions, which are **constant** on the dyadic intervals $[k2^{-\ell}, (k+1)2^{-\ell}[$, **converge** to u. Using that $\varphi_{n,k}$ is continuous we have

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We rewrite the above limit as a telescoping sum

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Using various estimates we obtain

$$|u((k+1)2^{-n}) - u(k2^{-n})|_H \leq C2^{-n}|u(k2^{-n})|_H \ln(1/|u(k2^{-n})|_H).$$

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Lemma (Discrete log-type Grownwall inequality)

Let $0 < \beta_0, \ldots, \beta_{2^n} < 1$ *and assume that*

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holds for all $k \in \{0, \ldots, 2^n - 1\}$. *Then, we have*

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Strong existence

The reduction via Girsanov transformation only works if our filtered probability space is equipped with a solution (which is $\mathcal{F}_t$-measurable).

Theorem

Given **any** *filtered probability space* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ *there is a path-by-path unique solution.*

On any filtered probability space we can prove that

$$(X, W),\ (Y, W) \text{ solutions} \qquad \Longrightarrow \qquad X = Y.$$

Hence, by Yamada–Watanabe we obtain a strong solution.

Any filtered probability space is therefore equipped with a solution, hence we can invoke our main result.

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Thank you for your attention!